LAMINAR FLOW IN A NARROW CYLINDRICAL TUBE

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Laminar flows during periodic pressure drops and transverse oscillations of the liquid-solid interface are discussed.

It is necessary to know the value of laminar flow velocity in the solution of certain applied problems. Among them are the secondary flows arising in tubing during transverse oscillations and the instability of nonresonance vibrational heating in infrared heaters.

In work on the ultrasonic capillary effect [1, 2], it was found that during contact of the ultrasonic radiator surface, which was submerged in a liquid, with the end of a capillary a flow rate in the capillary of the order of 10^{-1} m/sec was observed and the velocity was an order of magnitude less without contact; an increase in the frequency of the sound source above 20 kHz had no effect on the flow rate, and the maximum velocities were observed in capillaries with radii of 0.35-0.4 mm.

The data cited provides a basis for explanation by means of models which consider the flow of an incompressible fluid in a rigid cylindrical tube under the action of periodic forces and models which consider sonic flows produced by transverse oscillations of the channel walls.

We consider the cylindrical tube to be narrow in the acoustic sense ($R \ll \lambda$).

Model I. We assume the desired velocity distribution is independent of the x coordinate, along which the tube axis is directed. In this case, the convective terms in the equation of motion vanish, and the equation of nonstationary laminar flow takes the form [3]

$$\frac{\partial v}{\partial t} - \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) = f(t). \tag{1}$$

The boundary condition for Eq. (1) corresponds to adhesion of the liquid to the wall,

$$v(R,\ t) = 0. \tag{2}$$

At the initial time, the liquid is at rest,

$$v(r,\ 0) = 0. \tag{3}$$

If a one-to-one correspondence is established between the following parameters (the so-called electroacoustic analogy [4]), electric admittance A and acoustic admittance A, voltage E and force f, current I and velocity v, the liquid flow velocity under the action of an arbitrary force is described by an "acoustic Ohm's Law" given in the form of a contraction of force and total admittance,

$$v = f * A. \tag{4}$$

Performing a Laplace transformation of Eq. (1) with respect to the variable t and using the expansion theorem [5] for pulsating flow where $f(t) \sim \mathrm{Re}(f_0 e^{i\omega t})$, we obtain

$$v(r,\ t) = \mathrm{Re}\left\{f_0\left[d_1 e^{i\omega t} + \sum_{k=1}^{\infty} d_k e^{a_k t}\right]\right\},$$

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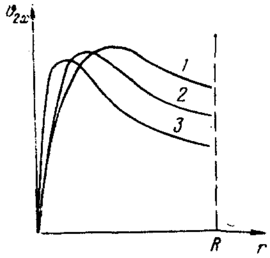

Fig. 1. Velocity profile for secondary acoustic flow in a channel with $R = 0.75 \cdot 10^{-3}$ m: 1) $\omega = 1265\ \sec^{-1}$; 2) $\omega = 3126\ \sec^{-1}$; 3) $\omega = 4366\ \sec^{-1}$.

where

$$d_k e^{a_k t} = \exp\left(-\frac{\lambda_k^2}{R^2}\nu t\right)\frac{J_0\left(\lambda_k \frac{r}{R}\right)}{\left[1+\left(\frac{\omega R^2}{\lambda_k^2 \nu}\right)^2\right]\lambda_k J_1(\lambda_k)}$$

and transforms to an expression corresponding to the response of the system (1)-(3) to a δ-shaped perturbation when $\omega < R^2/\lambda_k^2\nu$; at high frequencies, this term makes no significant contribution to the determination of the velocity,

$$d_1 = -\frac{i}{\omega}\left[1 - \frac{J_0\left(r\sqrt{i\frac{\omega}{\nu}}\right)}{J_0\left(R\sqrt{i\frac{\omega}{\nu}}\right)}\right]. \tag{5}$$

For $R\sqrt{\omega/\nu} \gg 1$, the time-averaged square of the velocity determined by means of d_1 has a maximum which is not located on the tube axis but is determined by the relation

$$(R - r_{\max})\sqrt{\frac{\omega}{2\nu}} \simeq 2{,}8.$$

This phenomenon was observed experimentally by E. Richardson and E. Taylor [6].

Knowing the average admittance over a cross section,

$$\overline{A}(t) = \frac{2}{R^2}\int_0^R rA(r,\ t)\,dr. \tag{6}$$

the average velocity over a cross section can be obtained for a narrow tube. Assuming $R\sqrt{\omega/\nu} > 10$ and using the approximate expression [7]

$$\frac{J_1(xi^{1/2})}{J_0(xi^{1/2})} \approx i,$$

we obtain the average admittance over a cross section on the basis of Eqs. (5) and (6),

$$\overline{A} = \frac{\delta_{ak}}{R}\left[1 + i\left(\frac{R}{\delta_{ak}} - 1\right)\right]. \tag{7}$$

For a pressure gradient of $10^6\ \mathrm{N/m^3}$ with $\omega \sim 10^5\ \sec^{-1}$ and $\nu \sim 10^{-6}\ \mathrm{m^2/sec}$, the flow velocity calculated from Eqs. (4) and (7) does not exceed 10^{-2} m/sec.

Thus the observed flows [1, 2] were not caused by a periodic pressure drop at the ends of a capillary created by an ultrasonic source.

Model II. We assume that the flows arise through the action of fields concentrated at the liquid-solid boundary. For both standing waves and for an oscillating cylinder, the properties of these flows are manifest near the boundary at a distance of the order of the thickness of the acoustic boundary layer, $\delta_{ak} = \beta^{-1}(\sqrt{2\nu})/\omega$. In the present case, ω is one of the normal oscillational frequencies of the capillary. Because of the complexity of the determination of boundary conditions for flows in a narrow cylindrical tube undergoing transverse oscillations, we consider the flows produced in a plane channel by transverse oscillations of its boundaries.

Excitation of a glass capillary by an ultrasonic source occurs in a limited frequency spectrum resulting from the inertial properties of the capillary which lead to considerable damping of normal oscillations when $\omega > 10^4\ \sec^{-1}$. An increase in the frequency of the exciting sonic source above 20 kHz at constant power has no significant effect on the oscillational spectrum of the capillary.

It was pointed out [8] that when $\beta h > 5$, the boundary at $z = 2h$ has no effect on flow near the test portion of the boundary at $z = 0$.

In the approximation

$$\rho = \rho_0 + \rho_1 e^{i\omega t},\quad P = P_0 + P_1 e^{i\omega t},\quad \overline{V} = \overline{V}_1 e^{i\omega t} + \overline{V}_2$$

TABLE 1. Values of Secondary Flow Parameters

ω, sec^{-1}	$(R-r_m)\cdot 10^{-4}$, m	β, m^{-1}	$\beta(R-r_m)$
1265	0,6	25152	1,5
3126	0,375	39534	1,5
4366	0,3	46723	1,4

we obtain equations for the rotational portion

$$2\overline{V}_{1c} = i\beta^{-2}\mathrm{rot\,rot}\overline{V}_{1c}, \tag{8}$$

and for the irrotational portion when $\beta \gg \omega/c$

$$2\overline{V}_{1b} = -i\beta^{-2}\,\mathrm{graddiv}\,\overline{V}_{1b}, \tag{9}$$

from the Navier–Stokes equation, the continuity equation, and the adiabatic relation which connects the values of P_1 and ρ_1, having represented $\overline{V}_1$ in the form of the sum

$$\overline{V}_1 = \overline{V}_{1c} + \overline{V}_{1b},$$

where

$$\mathrm{div}\,\overline{V}_{1c} = 0, \quad \mathrm{rot}\,\overline{V}_{1b} = 0,$$

On the basis of Eqs. (8) and (9), we obtain as in [8]

$$\overline{V}_1 = \left\{v_0(1-e^{-\Gamma z});\ 0;\ \left[-\frac{1}{k}\frac{\partial v_0}{\partial x} + \Gamma^{-1}\frac{\partial v_0}{\partial x}(1-e^{-\Gamma z})\right]\right\}, \tag{10}$$

for the plane channel, where

$$\Gamma = (1-i)\beta.$$

The component v_{1x} is a viscous Stokes wave attached to the boundary and the normal component v_{1z} at $z = 0$ is equal to the normal velocity of the boundary,

$$v_{1z}\Big|_{z=0} = \frac{\partial \xi}{\partial t}\Big|_{z=0}.$$

We obtain the tangential component of secondary flow velocity from

$$\nu\frac{\partial^2 v_{2x}}{\partial z^2} = \left\langle v_{1x}\frac{\partial v_{1x}}{\partial x}\right\rangle + \left\langle v_{1z}\frac{\partial v_{1x}}{\partial z}\right\rangle, \tag{11}$$

where the symbol $\langle\rangle$ denotes averaging over a cycle.

On the basis of Eq. (10), we have

$$\left\langle v_{1x}\frac{\partial v_{1x}}{\partial x}\right\rangle = \frac{\omega^{-1}}{2}v_0\frac{\partial v_0}{\partial x}(1-2C+e^{-2\beta z}),$$

$$\left\langle v_{1z}\frac{\partial v_{1x}}{\partial z}\right\rangle = \frac{\omega^{-1}}{2}v_0\frac{\partial v_0}{\partial x}\left[S-\frac{\beta}{k}(C+S)\right],$$

where

$$C = e^{-\beta z}\cos\beta z, \quad S = e^{-\beta z}\sin\beta z.$$

For the boundary conditions

$$v_{2x}|_{z=0} = 0, \quad \frac{\partial v_{2x}}{\partial z}\Big|_{z=h} = 0$$

we obtain from Eq. (11)

$$v_{2x} = -\frac{\omega^{-1}}{2}v_0\frac{\partial v_0}{\partial x}\Big(Az-\beta^2z^2+\frac{1}{2}E-2S+ + \frac{\beta}{k}(C-S-1)+1-C\Big), \tag{12}$$

where

$$E = (1 - e^{-2\beta z}),$$

$$A = 2\beta^2 h + \beta \left[3C(h) - 5S(h) + 2\frac{\beta}{k} C(h) - e^{-2\beta z} \right].$$

It is clear from Eq. (12) that the tangential component v_{2x} of the secondary flow is directed toward reduction of v_0 (immobilization of the capillary), as has been observed [1, 2].

The profile of v_{2x} in the channel is shown in Fig. 1 for $R = 0.75 \cdot 10^{-3}$ m and it is of interest that the maximum v_{2x} is separated from the boundary by a distance satisfying the relation (Table 1)

$$\beta (R - r_{max}) \simeq 1{,}5.$$

Calculations based on Eq. (12) for $k \sim 10^{-1}$ m, $\xi_0 \sim 10^{-5}$ m, and $R \sim 10^{-3}$ m give values for secondary flow velocity of the order of 10^{-2}-10^{-1} m/sec. A calculation of secondary flows averaged over cross section shows a tendency toward reduction in velocity with increase in capillary radius from 0.4 to 1 mm, which results from the reduction in the contribution of acoustic flows.

NOTATION

t	is the time;
x, z	are the coordinates;
r	is the current radius;
R	is the internal radius of tube;
ν	is the kinematic viscosity;
P_0, ρ_0	are the equilibrium pressure density;
P_1, ρ_1	are the variation of pressure, density in sonic wave;
λ_k	is the zeroes of Bessel function of zeroth order;
$2h$	is the height of channel;
k	is the wave number of respective boundary mode;
ξ_0	is the transverse displacement of boundary;
λ, c	are the wavelength and sound speed in liquid.

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